Triple Integrals in Rectangular Coordinates

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Overview

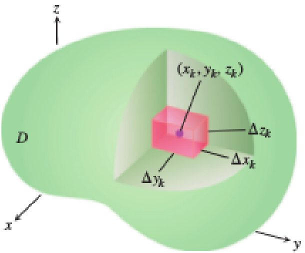
We use triple integrals to calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three-dimensional region.

Triple integrals also arise in the study of vector fields and fluid flow in three dimensions.

In the lecture, we discuss how to set up triple integrals and evaluate them.

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball, then the integral of F over D may be defined in the following way.

We partition a rectangular box-like region containing D into rectangular cells by planes parallel to the coordinate axis.



We number the cells that lie inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k.$$

We choose a point (x_k, y_k, z_k) in each cell and from the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k) \Delta V_k.$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that $\Delta x_k, \Delta y_k, \Delta z_k$ and the norm of the partition $\|P\|$, the largest value among $\Delta x_k, \Delta y_k, \Delta z_k$, all approach zero.

When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D.

Triple Integrals

It can be shown that when F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable.

As $\|P\| \to 0$ and the number of cells n goes to ∞, the sums S_n approach a limit. We call this limit the **triple integral of** F **over** D and write

$$\lim_{n\to\infty} S_n = \iiint\limits_D F(x,y,z)\, dV \quad \text{or} \quad \lim_{\|P\|\to 0} S_n = \iiint\limits_D F(x,y,z)\, dx\, dy\, dz.$$

The regions D over which continuous functions are integrable are those that can be closely approximated by small rectangular cells.

Such regions include those encountered in applications.

Volume of a Region in Space

If F is the constant function whose value is 1, then the sums reduce to

$$S_n = \sum F(x_k, y_k, z_k)\Delta V_k = \sum 1.\Delta V_k = \sum \Delta V_k.$$

As $\Delta x_k, \Delta y_k$ and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D.

We therefore define the volume of D to be the triple integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

Definition 1.

The volume *of a closed, bounded region D in space is*

$$V = \iiint_D dV.$$

Finding Limits of Integration

We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem to evaluate it by three repeated single integration.

As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

To evaluate

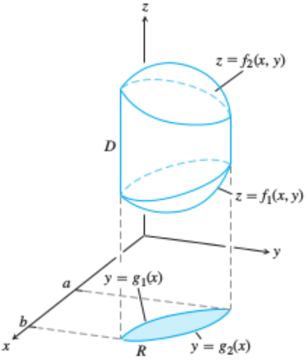
$$\iiint\limits_D F(x, y, z)\ dV$$

over a region D, integrate first with respact to z, then with respect to y, finally with x.

Finding Limits of Integration : Sketch

Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane.

Label the upper and lower bounding surfaces of D and the upper and the lower bounding curves of R.

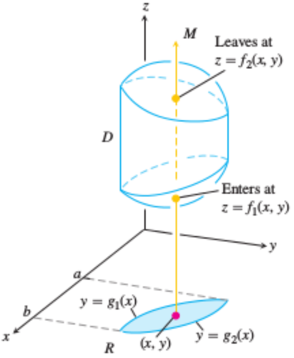


Finding Limits of Integration : z-Limits of Integration

Draw a line M passing through a typical point (x, y, z) in R parallel to the z-axis.

As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$.

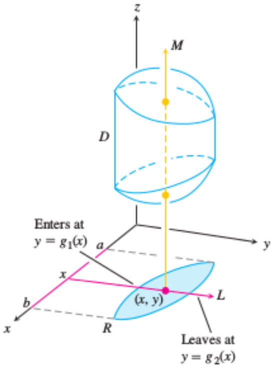
These are the z-limits of integration.



Finding Limits of Integration : y-Limits of Integration

Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.

These are the y-limits of integration.

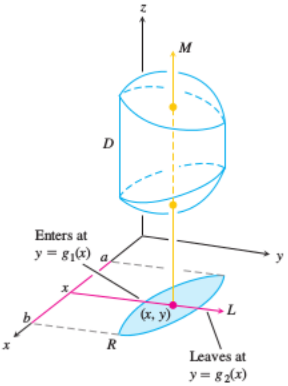


Choose x-limts that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the graph)

These are the x-limits of integration.

The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\ dz\ dy\ dx.$$



Follow similar procedures if you change the order of integration.

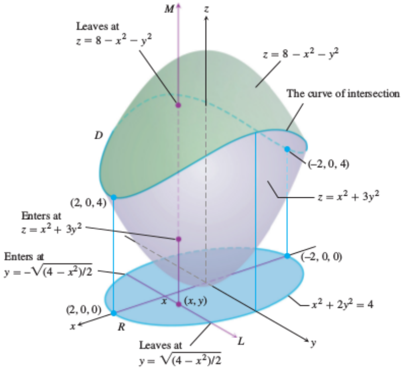
The "shadow' of region D lies in the plane or the last two variables with respect to which the iterated integration takes place.

The above procedure applies whenever a solid region D is bounded above and below by a surface, and when the "shadow" region R is bounded by a lower and upper curve.

It does not apply to regions with complicated holes through them, although sometimes such regions can be subdivided into simpler regions for which the procedure does apply.

Example 2.

The volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$ is calculated as follows.



The volume is

$$\iiint\limits_D dz\ dy\ dx,$$

the integral of $F(x, y, z) = 1$ over D.

To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces intersect on the elliptical cylinder

$$x^2 + 3y^2 = 8 - x^2 - y^2 \quad \text{of} \quad x^2 + 2y^2 = 4, \quad z > 0.$$

The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation : $x^2 + 2y^2 = 4$.

The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = \sqrt{(4 - x^2)/2}$.

Triple Integrals in Rectangular Coordinates - An Example

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axiz enters D at $z = x^2 + 3y^2$ leaves at $z = 8 - x^2 - y^2$.

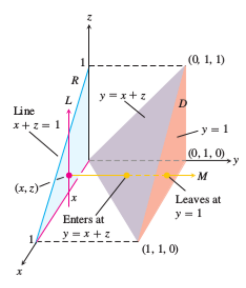
Next we find the y-limts of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$V = \iiint_D dz\ dy\ dx = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\ dy\ dx = 8\pi\sqrt{2}.$$

Example 3.

Let D be the tetrahedron D with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0)$, and $(0, 1, 1)$. We project D onto the xz-plane and set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over D. Here the order of integration is $dy\ dz\ dx$.



The integrals is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\ dy\ dz\ dx.$$

Six Ways of Ordering dx, dy, dz

As we have seen, there are sometimes (but not always) two different orders in which the iterated single integrations for evaluating a double integral may be worked.

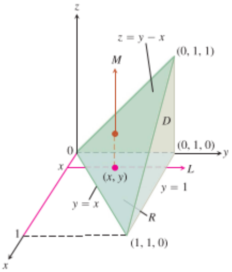
For triple integrals, there can be as many as six, since there are six ways of ordering dx, dy, and dz.

Each ordering leads to a different description of the region of integration in space, and to different limits of integration.

Example 4.

Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 3 in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution : First we find the z-limits of integration. A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$.



Next we find the y-limits of integration. On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$.

A line L through (x, y) parallel to the y-axis enters the shadow in the $xy-$plane at $y = x$ and exists at $y = 1$.

Solution (contd...)

Finally we find the $x-$limits of integration. As the line L parallel to the y-axis in the previous step seeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$.

The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\ dy\ dx.$$

For example, if $F(x, y, z) = 1$. we would find the volume of the tetrahedron to be

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\ dy\ dx = \frac{1}{6}.$$

We get the same result by integrating with the order $dy\ dz\ dx$.

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\ dz\ dx = \frac{1}{6}.$$

Average Value of Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint\limits_D F \, dV.$$

For example, if

$$F(x, y, z) = \sqrt{x^2 + y^2 + z^2},$$

then the average value of F over D is the average distance of points in D from the origin.

If $F(x, y, z)$ is the temperature at (x, y, z) on a solid that occupies a region D in space, then the average value of F over D is the average temperature of the solid.

Example 5.

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the panes $x = 2$, $y = 2$, and $x = 2$ in the first octant.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2 \int_0^2 \int_0^2 xyz \; dx \; dy \; dz = 8.$$

Hence

$$\begin{matrix} \text{Average value of} \\ xyz \text{ over the cube} \end{matrix} = \frac{1}{\text{volume}} \iiint_{cube} xyz \; dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx \; dy \; dz$, but any of the other five possible orders would have done as well.

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals.

If $F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

- Constant Multiple :

$$\iiint\limits_D kF\ dV = k \iiint\limits_D F\ dV \quad \text{(any number } k\text{)}$$

- Sum and Difference :

$$\iiint\limits_D (F \pm G)\ dV = \iiint\limits_D F\ dV \pm \iiint\limits_D G\ dV$$

Properties of Triple Integrals

- Domination :
  - $\iiint\limits_D F\ dV \geq 0 \quad \text{for} \ \ F \geq 0 \text{ on } D$
  - $\iiint\limits_D F\ dV \geq \iiint\limits_D G\ dV \quad \text{for} \ \ F \geq G \text{ on } D$
- Additivity :

$$\iiint\limits_D F\ dV = \iiint\limits_{D_1} F\ dV + \iiint\limits_{D_2} F\ dV$$

if D is the union of two nonoverlapping regions D_1 and D_2.

Exercise 6.

1. Volume of rectangular solid : *Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes* $x = 1, y = 2$, *and* $z = 3$. *Evaluate one of the integrals.*
2. Volume of tetrahedron : *Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane* $6x + 3y + 2z = 6$. *Evaluate one of the integrals.*
3. Volume of solid : *Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder* $x^2 + z^2 = 4$ *and the plane* $y = 3$. *Evaluate one of the integrals.*

Solution for (1.) in Exercise 6

$$\int_0^2\int_0^1\int_0^3 dz\,dx\,dy, \quad \int_0^3\int_0^2\int_0^1 dx\,dy\,dz,$$

$$\int_0^2\int_0^3\int_0^1 dx\,dz\,dy, \quad \int_0^3\int_0^1\int_0^2 dy\,dx\,dz,$$

$$\int_0^1\int_0^3\int_0^2 dy\,dz\,dx,$$

$$\int_0^1\int_0^2\int_0^3 dz\,dy\,dx = \int_0^1\int_0^2 3dy\,dx = \int_0^1 6\,dx = 6.$$

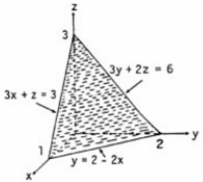
Solution for (2.) in Exercise 6

$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz\,dx\,dy$, $\quad \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy\,dz\,dx$,

$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy\,dx\,dz$, $\quad \int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx\,dz\,dy$,

$\int_0^3 \int_0^{2-2x/3} \int_0^{1-y/2-z/3} dx\,dy\,dz$,

$$\begin{aligned}
\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\,dy\,dx &= \int_0^1 \int_0^{2-2x} (3-3x-\frac{3}{2}y)dy\,dx \\
&= \int_0^1 [3(1-x)\cdot 2(1-x) - \frac{3}{4}\cdot 4(1-x)^2]dx \\
&= 3\int_0^1 (1-x^2)dx = [-(1-x)^3]_0^1 = 1.
\end{aligned}$$



$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz\, dx\, dy$, $\quad \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dy\, dz\, dx$,

$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 dy\, dx\, dz$, $\quad \int_0^2 \int_0^3 \int_0^{\sqrt{4-z^2}} dx\, dy\, dz$,

$\int_0^3 \int_0^2 \int_0^{\sqrt{4-z^2}} dx\, dz\, dy$,

$$\begin{aligned}
\int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dz\, dy\, dx &= \int_0^2 \int_0^3 \sqrt{4-x^2}\, dy\, dx \\
&= \int_0^2 3\sqrt{4-x^2}\, dx \\
&= \frac{3}{2}\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_0^2 \\
&= 6\sin^{-1} 1 \\
&= 3\pi.
\end{aligned}$$

Exercise 7.

1. Volume enclosed by paraboloids : *Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.*
2. Volume inside paraboloid beneath a plane : *Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\ dx\ dy$ and $dz\ dy\ dx$ that give the volume of D. Do not evaluate either integrals.*

Solution for (1.) in Exercise 7

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{x-x^2-y^2} dz\,dx\,dy,$$

$$\int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^{2}\int_{4}^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy,$$

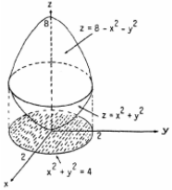
$$\int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz,$$

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^{2}\int_{4}^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx,$$

$$\int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$$

Solution for (1.) in Exercise 7 (contd...)

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx &= 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{x-x^2-y^2} dz\,dy\,dx \\
&= 4\int_0^2\int_0^{\sqrt{4-x^2}} [8-2(x^2+y^2)]dy\,dx \\
&= 8\int_0^2\int_0^{\sqrt{4-x^2}} (4-x^2-y^2)dy\,dx \\
&= 8\int_0^{\pi/2}\int_0^2 (4-r^2)r\,dr\,d\theta = 8\int_0^{\pi/2}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta \\
&= 32\int_0^{\pi/2} d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi
\end{aligned}$$



Solution for (2.) in Exercise 7

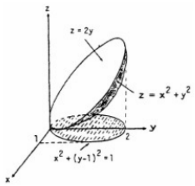
The projection of D onto the xy-plane has the boundary $x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$, which is a circle.

Therefore the two integrals are:

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\, dx\, dy$$

and

$$\int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1+x^2}} \int_{x^2+y^2}^{2y} dz\, dy\, dx.$$



Exercise 8.

Evaluate the following iterated integrals.

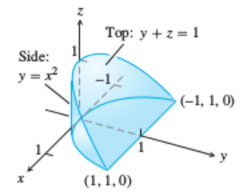
1. $\int_0^1 \int_0^1 \int_0^1 \left(x^2 + y^2 + z^2\right) \, dz \, dy \, dx$
2. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \, dy \, dx$
3. $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx \, dy \, dz$
4. $\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (x + y + z) \, dy \, dx \, dz$
5. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz \, dy \, dx$
6. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(u + v + w) \, du \, dv \, dw \quad (uvw\text{-space})$
7. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e se^s \ln r \frac{(\ln t)^2}{t} dt \, dr \, ds \quad (rst\text{-space})$
8. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx \, dt \, dv \; (tvx\text{-space})$
9. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} dp \, dq \, dr \; (pqr\text{-space})$

Solution for the Exercise 8

1. $\int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)dz\,dy\,dx = \int_0^1 \int_0^1 (x^2+y^2+\frac{1}{3})dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right)dx = 1$
2. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\,dy\,dx = \int_0^1 \int_0^{3-3x} (3-3x-y)dy\,dx =$
   $\int_0^1 \left[(3-3x)^2 - \frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1 (1-x^2)dx = -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$
3. $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z dx\,dy\,dz = \int_0^{\pi/6} \int_0^1 5y \sin z dy\,dz = \frac{5}{2}\int_0^{\pi/6} \sin z dz = \frac{5\left(2-\sqrt{3}\right)}{4}$
4. $\int_{-1}^1 \int_0^1 \int_0^2 (x+y+z)\,dy\,dx\,dz = \int_{-1}^1 \int_0^1 [xy+\frac{1}{2}y^2+zy]_0^2\,dx\,dz =$
   $\int_{-1}^1 \int_0^1 (2x+2+2z)dx\,dz = \int_{-1}^1 [x^2+2x+2zx]_0^1 dz = \int_{-1}^1 (3+2z)\,dz = [3z+z^2]_{-1}^1 = 6$
5. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}dy\,dx = \int_0^3 (9-x^2)\,dx = 18$
6. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)du\,dv\,dw = \int_0^\pi \int_0^\pi [\sin(w+v+\pi) - \sin(w+v)]dv\,dw =$
   $\int_0^\pi [(-\cos(w+2\pi)+\cos(w+\pi)) + (\cos(w+\pi)-\cos w)]dw =$
   $[-\sin(w+2\pi)+\sin(w+\pi)-\sin w+\sin(w+\pi)]_0^\pi = 0$
7. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e se^s \ln r \frac{(\ln t)^2}{t} dt\,dr\,ds = \int_0^1 \int_1^{\sqrt{e}} (se^s \ln r) \left[\frac{1}{3}(\ln t)^3\right]_1^e dr\,ds =$
   $\int_0^1 \int_1^{\sqrt{e}} \frac{se^s}{3} \ln r\,dr\,ds = \int_0^1 \frac{se^s}{3}[r\ln r - r]_1^{\sqrt{e}} ds = \frac{2-\sqrt{e}}{6}\int_0^1 se^s\,ds = \frac{2-\sqrt{e}}{6}[se^s - e^s]_0^1 = \frac{2-\sqrt{e}}{6}$
8. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-x}^{2t} e^x\,dx\,dt\,dv = \int_0^{\pi/4} \int_0^{\ln \sec v} \lim_{b \to -\infty} (e^{2t} - e^b)dt\,dv =$
   $\int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t}\,dt\,dv = \int_0^{\pi/4} \left(\frac{1}{2}e^{2\ln \sec v} - \frac{1}{2}\right) dv = \left[\frac{\tan v}{2} - \frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2} - \frac{\pi}{8}$
9. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} dp\,dq\,dr = \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1} dq\,dr = \int_0^7 \frac{1}{3(r+1)}\left[-(4-q^2)^{3/2}\right]_0^2 dr =$
   $\frac{8}{3}\int_0^7 \frac{1}{r+1} dr = \frac{8\ln 8}{3} = 8\ln 2$

Exercise 9.

Here is the region of integration of the integral $\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\, dy\, dx.$



Rewrite the integral as an equivalent iterated integral in the order

(a) $dy\ dz\ dx$

(b) $dx\ dy\ dz$

(c) $dy\ dx\ dz$

(d) $dx\ dz\ dy$

Solution for the Exercise 9

(a) $\displaystyle\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$

(b) $\displaystyle\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$

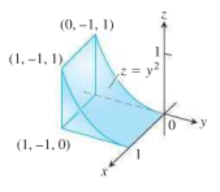
(c) $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$

(d) $\displaystyle\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$

Exercise 10.

Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\ dy\ dx.$$



Rewrite the integral as an equivalent iterated integral in the order

(a) $dy\ dz\ dx$
(b) $dy\ dx\ dz$
(c) $dx\ dy\ dz$
(d) $dx\ dz\ dy$
(e) $dz\ dx\ dy$.

Solution for the Exercise 10

(a) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$

(b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$

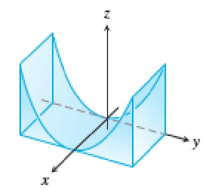
(c) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\, dy\, dz$

(d) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\, dz\, dy$

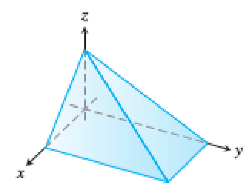
(e) $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\, dx\, dy$

Exercise 11.

1. *The region between the cylinder* $z = y^2$ *and the xy-plane that is bounded by the planes* $x = 0, x = 1, y = -1, y = 1$.



2. *The region in the first octant bounded by the coordinate planes and the planes* $x + z = 1, y + 2z = 2$.

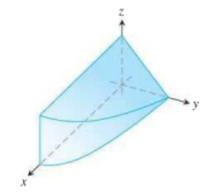


Solution for the Exercise 11

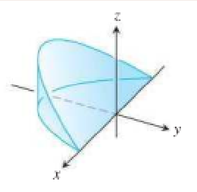
1. $V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\, dy\, dx = \int_0^1 \int_{-1}^1 y^2 dy\, dx = \frac{2}{3} \int_0^1 dx = \frac{2}{3}$
2. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2x} dy\, dz\, dx = \int_0^1 \int_0^{1-x} (2-2z) dz\, dx = \int_0^1 [2z - z^2]_0^{1-x} dx =$
   $\int_0^1 (1-x^2) dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

Exercise 12.

1. *The region in the first octant bounded by the coordinate planes, the plane* $y + z = 2$*, and the cylinder* $x = 4 - y^2$*.*



2. *The wedge cut from the cylinder* $x^2 + y^2 = 1$ *by the planes* $z = -y$ *and* $z = 0$*.*



Solution for the Exercise 12

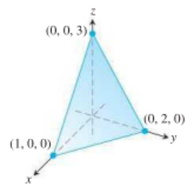
1.

$$
\begin{aligned}
V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx &= \int_0^4 \int_0^{\sqrt{4-x}} (2-y) dy\, dx \\
&= \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2} \right) \right] dx \\
&= \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2 \right]_0^4 = \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) \\
&= \frac{32}{3} - 4 = \frac{20}{3}
\end{aligned}
$$

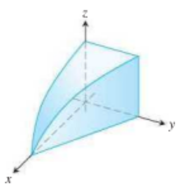
2. $V = 2\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\, dy\, dx = -2\int_0^1 \int_{-\sqrt{1-x^2}}^0 y\, dy\, dx = \int_0^1 (1-x^2) dx = \frac{2}{3}$

Exercise 13.

1. *The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through* $(1, 0, 0)$, $(0, 2, 0)$, *and* $(0, 0, 3)$.



2. *The region in the first octant bounded by the coordinate planes, the planes* $y = 1 - x$, *and the surface* $z = \cos(\pi x/2)$, $0 \leq x \leq 1$.



Solution for the Exercise 13

1.

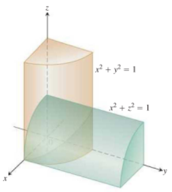
$$V=\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx = \int_0^1\int_0^{2-2x}(3-3x-\frac{3}{2}y)dy\,dx$$
$$= \int_0^1\left[6(1-x^2)-\frac{3}{4}\cdot 4(1-x)^2\right]dx$$
$$= \int_0^1 3(1-x)^2dx = \left[-(1-x)^3\right]_0^1 = 1$$

2.

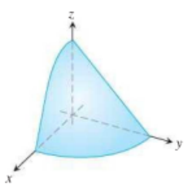
$$V=\int_0^1\int_0^{1-x}\int_0^{\cos[\pi x/2]} dz\,dy\,dx = \int_0^1\int_0^{1-x}\cos(\frac{\pi x}{2})dy\,dx = \int_0^1\left(\cos\frac{\pi x}{2}\right)(1-x)\,dx$$
$$= \int_0^1\cos\left(\frac{\pi x}{2}\right)dx - \int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$
$$= \left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_0^1 - \frac{4}{\pi^2}\int_0^{\pi/2} u\cos u\,du$$
$$= \frac{2}{\pi}-\frac{4}{\pi^2}[\cos u+u\sin u]_0^{\pi/2} = \frac{2}{\pi}-\frac{4}{\pi^2}(\frac{\pi}{2}-1) = \frac{4}{\pi^2}$$

Exercise 14.

1. *The region common to the interiors of the cylinders* $x^2 + y^2 = 1$ *and* $x^2 + z^2 = 1$*, one-eight of which is shown in the accompanying figure.*



2. *The region in the first octant bounded by the coordinate planes and the surface* $z = 4 - x^2 - y$.



Solution for the Exercise 14

1.

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$$

$$= 8\int_0^1 (1-x^2)\,dx = \frac{16}{3}$$

2.

$$V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx = \int_0^2 \int_0^{4-x^2} (4-x^2-y)\,dy\,dx$$

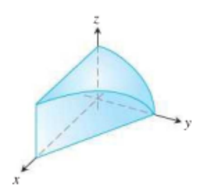
$$= \int_0^2 \left[(4-x^2)^2 - \frac{1}{2}(4-x^2)^2\right] dx$$

$$= \frac{1}{2}\int_0^2 (4-x^2)^2 dx$$

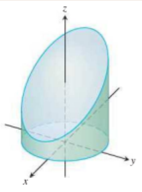
$$= \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \frac{128}{15}$$

Exercise 15.

1. *The region in the first octant bounded by the coordinate planes, the plane* $x + y = 4$*, and the cylinder* $y^2 + 4z^2 = 16$*.*



2. *The region cut from the cylinder* $x^2 + y^2 = 4$ *by the plane* $z = 0$ *and the plane* $x + z = 3$*.*



Solution for the Exercise 15

1. $V = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} \int_0^{4-y} dx\, dz\, dy = \int_0^4 \int_0^{(\sqrt{16-y^2})/2} (4-y)\, dz\, dy =$
$\int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\, dy = \int_0^4 2\sqrt{16-y^2}dy - \frac{1}{2}\int_0^4 y\sqrt{16-y^2}dy =$
$\left[y\sqrt{16-y^2} + 16\sin^{-1}\frac{y}{4}\right]_0^4 + \left[\frac{1}{6}(16-y^2)^{3/2}\right]_0^4 = 16(\frac{\pi}{2}) - \frac{1}{6}(16)^{3/2} = 8\pi - \frac{32}{3}$

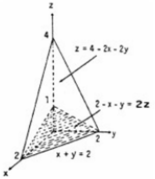
2. $V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz\, dy\, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) dy\, dx =$
$2\int_{-2}^2 (3-x)\sqrt{4-x^2}dx = 3\int_{-2}^2 2\sqrt{4-x^2}dx - 2\int_{-2}^2 x\sqrt{4-x^2}dx =$
$3\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[\frac{2}{3}(4-x^2)^{3/2}\right]_{-2}^2 = 12\sin^{-1}1 - 12\sin^{-1}(-1) =$
$12\left(\frac{\pi}{2}\right) - 12\left(-\frac{\pi}{2}\right) = 12\pi$

Exercise 16.

1. *The region between the planes* $x + y + 2z = 2$ *and* $2x + 2y + z = 4$ *in the first octant.*
2. *The finite region bounded by the planes* $z = x, x + z = 8$, $z = y$, $y = 8$, *and* $z = 0$.
3. *The region cut from the solid elliptical cylinder* $x^2 + 4y^2 \leq 4$ *by the xy-plane and plane* $z = x + 2$.
4. *the region bounded in back by the plane* $x = 0$, *on the front and sides by the parabolic cylinder* $x = 1 - y^2$, *on the top by the paraboloid* $z = x^2 + y^2$, *and on the bottom by the* xy*-plane.*

Solution for (1.) in Exercise 16

$$
\begin{aligned}
\int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx &= \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2}\right) dy\,dx \\
&= \int_0^2 \left[3\left(1 - \frac{x}{2}\right)(2-x) - \frac{3}{4}(2-x)^2\right] dx \\
&= \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4}\right] dx \\
&= \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4}\right]_0^2 \\
&= (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2
\end{aligned}
$$



Solution for (2.) in Exercise 16

$$
\begin{aligned}
V = \int_0^4 \int_z^8 \int_z^{8-z} dx\, dy\, dz &= \int_0^4 \int_z^8 (8-2z) dy\, dz \\
&= \int_0^4 (8-2z)(8-z) dz = \int_0^4 (64 - 24z + 2z^2) dz \\
&= \left[64z - 12z^2 + \frac{2}{3} z^3 \right]_0^4 = \frac{320}{3}
\end{aligned}
$$

Solution for (3.) in Exercise 16

$$\begin{aligned}
V = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}/2}\int_{0}^{x+2} dz\,dy\,dx &= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}/2}(x+2)dy\,dx \\
&= \int_{-2}^{2}(x+2)\sqrt{4-x^2}dx \\
&= \int_{-2}^{2}2\sqrt{4-x^2}dx + \int_{-2}^{2}x\sqrt{4-x^2}dx \\
&= \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2} + \left[-\frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^{2} \\
&= 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi
\end{aligned}$$

Solution for (4.) in Exercise 16

$$
\begin{aligned}
V = 2\int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz\, dx\, dy &= 2\int_0^1 \int_0^{1-y^2} (x^2+y^2) dx\, dy \\
&= 2\int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y^2} dy \\
&= 2\int_0^1 (1-y^2)\left[\frac{1}{3}(1-y^2)^2 + y^2\right] dy \\
&= 2\int_0^1 (1-y^2)\left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4\right) dy \\
&= \frac{2}{3}\int_0^1 (1-y^6) dy \\
&= \frac{2}{3}\left[y - \frac{y^7}{7}\right]_0^1 \\
&= \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = \frac{4}{7}
\end{aligned}
$$

Exercise 17.

In the following exercises, find the average value of $F(x, y, z)$ over the given region.

1. $F(x, y, z) = x^2 + 9$ *over the cube in the first octant bounded by the coordinate planes and the planes* $x = 2, y = 2$ *and* $z = 2$.
2. $F(x, y, z) = x + y - z$ *over the rectangular solid in the first octant bounded by the coordinate planes and the planes* $x = 1, y = 1$, *and* $z = 2$.
3. $F(x, y, z) = x^2 + y^2 + z^2$ *over the cube in the first octant bounded by the coordinate planes and the planes* $x = 1, y = 1$, *and* $z = 1$.
4. $F(x, y, z) = xyz$ *over the cube in the first octant bounded by the coordinate planes and the planes and the planes* $x = 2, y = 2$, *and* $z = 2$.

Solution for the Exercise 17

1. average$= \frac{1}{8}\int_0^2\int_0^2\int_0^2 (x^2+9)dz\,dy\,dx = \frac{1}{8}\int_0^2\int_0^2 (2x^2+18)dy\,dx = \frac{1}{8}\int_0^2 (4x^2+36)dx = \frac{31}{3}$
2. average$= \frac{1}{2}\int_0^1\int_0^1\int_0^2 (x+y-z)dz\,dy\,dx = \frac{1}{2}\int_0^1\int_0^1 (2x+2y-2)dy\,dx = \frac{1}{2}\int_0^1 (2x-1)dx = 0$
3. average$= \int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)dz\,dy\,dx = \int_0^1\int_0^1 (x^2+y^2+\frac{1}{3})dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$
4. average$= \frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz dz\,dy\,dx = \frac{1}{4}\int_0^2\int_0^2 xy\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = 1$

Exercise 18.

Evaluate the integrals in the following exercises, by changing the order of integration in an appropriate way.

1. $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos\left(x^2\right)}{2\sqrt{z}}\,dx\ dy\ dz$
2. $\displaystyle\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\,dy\ dx\ dz$
3. $\displaystyle\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\,dy\ dz\ dx$

Solution for the Exercise 18

1. $$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4\cos(x^2)}{2\sqrt{z}}\, dy\, dx\, dz =$$
$$\int_0^4 \int_0^2 \frac{x\cos(x^2)}{\sqrt{z}}\, dx\, dz = \int_0^4 \left(\frac{\sin 4}{2}\right) z^{-1/2} dz = \left[(\sin 4) z^{1/2}\right]_0^4 = 2\sin 4$$

2. $$\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\, dx\, dy\, dz = \int_0^1 \int_0^1 6yze^{zy^2}\, dy\, dz =$$
$$\int_0^1 \left[3e^{zy^2}\right]_0^1 dz = 3\int_0^1 (e^z - z) dz = 3\left[e^z - 1\right]_0^1 = 3e - 6$$

3. $$\int_0^2 \int_0^{4-x^2} \int_0^4 \frac{\sin 2z}{4-z}\, dy\, dz\, dx = \int_0^2 \int_0^{4-x^2} \frac{x\sin 2z}{4-z}\, dz\, dx =$$
$$\int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z}\right) x\, dx\, dz = \int_0^4 \left(\frac{\sin 2z}{4-z}\right) \frac{1}{2}(4-z) dz = \left[-\frac{1}{4}\cos 2z\right]_0^4 =$$
$$\left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$$

Exercise 19.

1. Finding an upper limit of an iterated integral : *Solve for a :*

$$\int_0^1 \int_0^{4-a-x^2} \int_0^{4-x^2-y} dz\ dy\ dx = \frac{4}{15}.$$

2. Ellipsoid : *For what value of c is the volume of the ellipsoid*

$$x^2 + (y/2)^2 + (z/c)^2 = 1$$

equal to 8π?

Solution for (1.) in Exercise 19

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\, dy\, dx = \frac{4}{15}$$

$$\Rightarrow \quad \int_0^1 \int_0^{4-a-x^2} (4 - x^2 - y - a)\, dy\, dx = \frac{4}{15}$$

$$\Rightarrow \quad \int_0^1 \left[(4 - a - x^2)^2 - \frac{1}{2}(4 - a - x^2)^2 \right] dx = \frac{4}{15}$$

$$\Rightarrow \quad \frac{1}{2} \int_0^1 (4 - a - x^2)^2\, dx = \frac{4}{15}$$

$$\Rightarrow \quad \int_0^1 [(4 - a)^2 - 2x^2(4 - a) + x^4] dx = \frac{8}{15}$$

$$\Rightarrow \quad \left[(4 - a)^2 x - \frac{2}{3}x^3(4 - a) + \frac{x^3}{5} \right]_0^1 = \frac{8}{15}$$

$$\Rightarrow \quad (4 - a)^2 - \frac{2}{3}(4 - a) + \frac{1}{5} = \frac{8}{15}$$

$$\Rightarrow \quad 15(4 - a)^2 - 10(4 - a) - 5 = 0$$

$$\Rightarrow \quad [3(4 - a) + 1][(4 - a) - 1] = 0$$

$$\Rightarrow \quad a = \frac{13}{3} \text{ or } a = 3.$$

Solution for (2.) in Exercise 19

The volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is

$$\frac{4abc\pi}{3}.$$

If the volume of the ellipsoid equals to 8π, then

$$\frac{4(1)(2)(c)\pi}{3} = 8\pi.$$

Hence $c = 3$.

Exercise 20.

1. Minimizing a triple integral *What domain D in space minimizes the value of the integral*

$$\iiint_D \left(4x^2 + 4y^2 + z^2 - 4\right) dV?$$

Give reasons for your answer.

2. Maximizing a triple integral : *What domain D in maximizes the value of the integral*

$$\iiint_D \left(1 - x^2 - y^2 - z^2\right) dV?$$

Give reasons for your answer.

Solution for the Exercise 20

1. The minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that

$$4x^2 + 4y^2 + z^2 - 4 \leq 0 \quad \text{or } 4x^2 + 4y^2 + z^2 \leq 4,$$

which is a solid ellipsoid centered at the origin.

2. To minimize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that

$$1 - x^2 - y^2 - z^2 \geq \quad 0 \text{ or } \quad x^2 + y^2 + z^2 \leq 1,$$

which is a solid sphere of radius 1 centered at the origin.

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